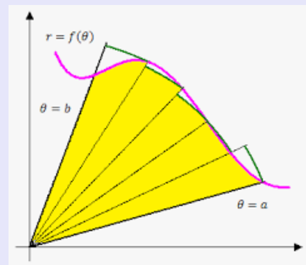
Calculus II

Lecture 25



Feb 19-8:47 AM

Class QZ 24

Determine whether $\sum_{n=1}^{\infty} \frac{n^3}{4^n}$ Converges or diverges.

Ratio test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{4^{n+1}}}{\frac{n^3}{4^n}}$

$= \lim_{n\to\infty} \frac{(n+1)^3 \cdot \cancel{4^n}}{n^3 \cdot \cancel{4^{n+1}}} = \lim_{n\to\infty} \frac{1}{4}\left(\frac{n+1}{n}\right)^3$

$\frac{n+1}{n} \to 1$

$= \frac{1}{4} \cdot 1 = \frac{1}{4} < 1$

By ratio Test

It converges.

Jul 24-7:01 AM

Class QZ 23

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ converges or diverges.

Alternating Series

$a_n = \frac{1}{\sqrt{n}}$

$\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$

1) $a_{n+1} < a_n$ ✓

2) $\lim_{n\to\infty} a_n = 0$ ✓

It converges by A.S.T.

Jul 23-9:58 AM

Find radius of convergence and interval of convergence for $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

Ratio Test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+1)^2+1}}{\frac{(x-2)^n}{n^2+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{\cancel{(x-2)^{n+1}}^{\,x-2}}{(n+1)^2+1} \cdot \frac{n^2+1}{\cancel{(x-2)^n}} \right| = |x-2| \boxed{\lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1}} \to 1$

$= |x-2| \cdot 1 = |x-2|$

By ratio Test, it converges when $L < 1$

$|x-2| < 1 \quad \to -1 < x-2 < 1$

$1 < x < 3$

$[\ 1 \quad 2 \quad 3\ ]$

Center of Conv. at 2

Radius 1

Try $x = 3$

$\sum_{n=0}^{\infty} \frac{(3-2)^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{1}{n^2+1}$

Converges by Comparison with P-Series $P=2$

Try $x = 1$

$\sum_{n=0}^{\infty} \frac{(1-2)^n}{n^2+1}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ Converges by A.S.T.

Interval of Convergence

$[1, 3]$

Jul 24-8:16 AM

$$\sum_{n=1}^{\infty} n!\,(2x-1)^n$$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\overset{n+1}{\cancel{(n+1)!}}\;\overset{2x-1}{\cancel{(2x-1)^{n+1}}}}{\cancel{n!}\;\cancel{(2x-1)^n}}\right|$$

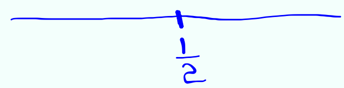
Converges

$L < 1$

$$= |2x-1|\cdot\boxed{\lim_{n\to\infty}(n+1)} \to \infty$$

$$|2x-1| = 0 \qquad \boxed{x=\tfrac{1}{2}}$$

Converges when $x=\frac{1}{2}$



Jul 24-8:26 AM

$$\sum_{n=1}^{\infty}\frac{n^2x^n}{2\cdot 4\cdot 6\cdots(2n)}$$

$$2\cdot 1\cdot 2\cdot 2\cdot 2\cdot 3\cdots 2\cdot n$$

$$2^n\cdot n!$$

Ratio Test

$$\sum_{n=1}^{\infty}\frac{n^2x^n}{2^n\cdot n!}$$

$$\lim_{n\to\infty}\left|\frac{(n+1)^2\cdot\cancel{x^{n+1}}}{\cancel{2^{n+1}}\cdot\underset{n+1}{\cancel{(n+1)!}}}\cdot\frac{\cancel{2^n}\,\cancel{n!}}{n^2\,\cancel{x^n}}\right|$$

$$=\frac{1}{2}\cdot|x|\cdot\lim_{n\to\infty}\frac{\cancel{(n+1)^2}}{\cancel{(n+1)}\,n^2}$$

$$=\frac{|x|}{2}\left\{\lim_{n\to\infty}\frac{n+1}{n^2}\right\}\to 0$$

$$=\frac{|x|}{2}\cdot 0 = 0 < 1$$

$$(-\infty,\infty)$$

Jul 24-8:32 AM

find a power series for $f(x)=\frac{1}{(1+x)^2}$

$$\frac{1}{(1+x)^2}=\frac{d}{dx}\left[\frac{-1}{1+x}\right]=\frac{0\cancel{(1+x)}^{\to 0}-(-1)\cdot 1}{(1+x)^2}=\frac{1}{(1+x)^2}$$

$$1+x+x^2+x^3+\cdots=\frac{1}{1-x}$$

$$\frac{d}{dx}\left[\frac{-1}{1+x}\right]=-1\frac{d}{dx}\left[\frac{1}{1-(-x)}\right]$$

$$=-\frac{d}{dx}\left[1-x+x^2-x^3+x^4\cdots\right]$$

$$=-\left[0-1+2x-3x^2+4x^3-\cdots\right]$$

$$=1-2x+3x^2-4x^3+\cdots$$

$$\frac{1}{(1+x)^2}=\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$$

Jul 24-8:42 AM

$$\int\frac{\tan^{-1}x}{x}\,dx$$

$$\tan^{-1}x=\int\frac{1}{1+x^2}\,dx$$

$$=\int\left[1-x^2+x^4-x^6+x^8\cdots\right]dx$$

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}+\cdots+C$$

$$\tan^{-1}0=0-0+0-0+\cdots+C$$

$$\boxed{C=0}$$

$$\frac{\tan^{-1}x}{x}=\frac{1}{x}\left[x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}+\cdots\right]$$

$$=\left[1-\frac{x^2}{3}+\frac{x^4}{5}-\frac{x^6}{7}+\frac{x^8}{9}+\cdots\right]$$

$$\int\frac{\tan^{-1}x}{x}\,dx=x-\frac{x^3}{3\cdot3}+\frac{x^5}{5\cdot5}-\frac{x^7}{7\cdot7}+\frac{x^9}{9\cdot9}\cdots$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)^2}+C$$

Jul 24-8:50 AM

Find Taylor Series for $f(x) = \ln x$ near 2.

$a = 2$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$f(x) = \ln x$ $\quad$ $f(2) = \ln 2$

$f'(x) = \frac{1}{x}$ $\quad$ $f'(2) = \frac{1}{2}$

$f''(x) = \frac{-1}{x^2}$ $\quad$ $f''(2) = \frac{-1}{4}$

$f'''(x) = \frac{2}{x^3}$ $\quad$ $f'''(2) = \frac{2}{2^3} = \frac{1}{4}$

Taylor Series

$$\ln 2 + \frac{1}{2}(x-2) + \frac{-\frac{1}{4}}{2!}(x-2)^2 + \frac{\frac{1}{4}}{3!}(x-2)^3 + \cdots$$

Jul 24-8:58 AM

Find Maclaurin Series for $f(x) = \sinh x$

Recall $\sinh x = \frac{e^x - e^{-x}}{2}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^x - e^{-x} = 2x + 2 \cdot \frac{x^3}{3!} + 2 \cdot \frac{x^5}{5!} + 2 \cdot \frac{x^7}{7!} + \cdots$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$$

Jul 24-9:03 AM

Find Maclaurin Series for $\cosh x$.

$\hookrightarrow \frac{e^x+e^{-x}}{2}$

Recall $\frac{d}{dx}[\sinh x] = \cosh x$

$$\cosh x = \frac{d}{dx}\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots\right]$$

$$= 1 + \frac{3x^2}{3!} + \frac{5x^4}{5!} + \frac{7x^6}{7!} + \frac{9x^8}{9!} + \cdots$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\frac{e^x+e^{-x}}{2} = \frac{2 + 2\cdot\frac{x^2}{2!} + 2\cdot\frac{x^4}{4!} + 2\cdot\frac{x^6}{6!} + \cdots}{2}$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!}$$

Jul 24-9:11 AM

Find First 4 non-zero terms of Maclaurin Series for $e^x \cos x$.

$$e^x \cos x = \left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right]\left[1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right]$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$$

$$x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^7}{720} + \cdots$$

$$\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{48} - \frac{x^8}{1440} + \cdots$$

$$\frac{x^3}{6} - \frac{x^5}{12} + \frac{x^7}{148} + \cdots$$

$$= \boxed{1 + x - \frac{1}{3}x^3 - \frac{5}{24}x^4 +} \cdots$$

Jul 24-9:19 AM

Find Maclaurin Series for $x \cos\left(\frac{x^2}{2}\right)$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos \frac{x^2}{2} = 1 - \frac{\left(\frac{x^2}{2}\right)^2}{2!} + \frac{\left(\frac{x^2}{2}\right)^4}{4!} - \frac{\left(\frac{x^2}{2}\right)^6}{6!} + \cdots$$

$$= 1 - \frac{x^{2\cdot 2}}{2^2 \cdot 2!} + \frac{x^{2\cdot 4}}{2^4 \cdot 4!} - \frac{x^{2\cdot 6}}{2^6 \cdot 6!}$$

$$x \cos \frac{x^2}{2} = x - \frac{x^{2\cdot 2+1}}{2^2 \cdot 2!} + \frac{x^{2\cdot 4+1}}{2^4 \cdot 4!} - \frac{x^{2\cdot 6+1}}{2^6 \cdot 6!}$$

$$= \sum_{n=0} \frac{(-1)^n x^{2\cdot 2n+1}}{2^{2n}(2n)!}$$

Jul 24-9:28 AM

Find Maclaurin Series for $\sin \pi x$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin \pi x = \pi x - \frac{\pi^3 x^3}{3!} + \frac{\pi^5 x^5}{5!} - \frac{\pi^7 x^7}{7!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (\pi x)^{2n-1}}{(2n-1)!}$$

Jul 24-9:36 AM

Binomial Expansion

$(A+B)^n \leftarrow$ whole #

$n=0$ 1

$n=1$ 1 1

$n=2$ 1 2 1

$n=3$ 1 3 3 1

$n=4$ 1 4 6 4 1

$$(A+B)^n = \binom{n}{0}A^n + \binom{n}{1}A^{n-1}B + \binom{n}{2}A^{n-2}B^2 + \binom{n}{3}A^{n-3}B^3$$

Binomial Coef. $\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!}$

$n=7$

$\binom{7}{0} = \frac{7!}{0!\cdot 7!} = 1$, $\binom{7}{1} = \frac{7!}{1!\cdot 6!} = 7$, $\binom{7}{2} = \frac{7!}{2!\cdot 5!} = 21$

$$(A+B)^7 = 1A^7 + 7A^6B + 21A^5B^2 + 35A^4B^3 + 35$$

$\binom{7}{3} = \frac{7!}{3!\cdot 4!}$

Jul 24-9:41 AM

n is not a whole #

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

$$\sqrt{1+x} = (1+x)^{1/2}$$

$\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2} = \frac{3}{8}$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{\frac{3}{8}}{3!}x^3 + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$$

Let $x = 1$

$$\sqrt{1+1} \approx 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16}$$

$$\sqrt{2} \approx \frac{23}{16} = 1.438$$

from calc. $\sqrt{2} \approx 1.414$

Jul 24-9:53 AM